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Harmonics on the quantum Euclidean space

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Abstract
We study harmonic polynomials on the quantum Euclidean space E_q^N generated by quantum coordinates $x_i, i = 1, 2, \ldots, N$, on which the quantum group $SO_q(N)$ acts. They are defined as solutions of the equation $\Delta_q p = 0$, where Δ_q is the q-Laplace operator on E_q^N. We construct a q-analogue of the classical zonal polynomials and associated spherical polynomials with respect to the quantum subgroup $SO_q(N-2)$. The associated spherical polynomials constitute an orthogonal basis of the spaces of homogeneous harmonic polynomials. They are represented as products of polynomials depending on q-radii and $x_j, x_{j'}, j' = N - j + 1$. This representation is, in fact, a q-analogue of the classical separation of variables.

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1. Introduction

The Laplace operator, harmonic polynomials and related separations of variables are of great importance in classical analysis. They are closely related to the rotation group $SO(N)$ and its subgroups; see, for example, [1], chapter 10.

Many new directions of contemporary mathematical physics are related to quantum groups and noncommutative geometry. It is natural to generalize the classical theory of harmonic polynomials to noncommutative case. Such a generalization can be of great importance for further development of some branches of mathematical and theoretical physics related to noncommutative geometry. For example, it can be used for the development of quantum mechanics and field theory on noncommutative spaces. In particular, the Laplace operators on the quantum spaces are useful for the construction of operators of quantum mechanics. It is interesting to generalize the results of this paper to the case of harmonic functions (in particular, to the case of functions with singularities).

The aim of this paper is to construct a q-deformation of many aspects of the classical theory of harmonic polynomials. In the q-case, instead of the Euclidean space we have the quantum Euclidean space. This is defined in terms of the associative algebra $\mathcal{A}$ generated by the noncommuting elements $x_1, x_2, \ldots, x_N$ satisfying the certain defining relations. These elements play the role of Cartesian coordinates of the classical Euclidean space E_N.

The q-Laplace operator Δ_q on $\mathcal{A}$ is defined in terms of q-derivatives (see formula (16)). Instead of the group $SO(N)$ we have the quantum group $SO_q(N)$ or the corresponding quantum algebra $U_q(\mathrm{so}_N)$. In our exposition, it is more convenient to deal with the algebra $U_q(\mathrm{so}_N)$. The q-harmonic polynomials on the quantum Euclidean space are defined as elements p of $\mathcal{A}$ (that is, polynomials in quantum coordinates $x_1, x_2, \ldots, x_N$) for which $\Delta_q p = 0$. We construct projectors $\mathsf{H}_m : \mathcal{A}_m \to \mathcal{H}_m$, where $\mathcal{A}_m$ and $\mathcal{H}_m$ are the subspaces of homogeneous (of degree m) polynomials in $\mathcal{A}$ and in the space $\mathcal{H}$ of all q-harmonic polynomials from $\mathcal{A}$, respectively. Using these projectors we construct in $\mathcal{H}_m$ a q-analogue of associated spherical harmonics with respect to the quantum subgroup $SO_q(N-2)$. They constitute an orthogonal basis of the space $\mathcal{H}_m$ corresponding to the chain of the quantum subgroups $SO_q(N) \supset SO_q(N-2) \supset SO_q(N-4) \supset \cdots \supset SO_q(3)$ (or $SO_q(2)$). Here we obtain a q-analogue of the corresponding separation of polyspherical coordinates. Our construction is similar that which we used in [2] for the case of quantum complex vector space with the quantum unitary group $U_q(N)$ as a quantum motion group. Our derivations use essentially the results of [3]. Note that our approach is different from that used in [4] since we use the projection technics described in section 5. Besides, the noncommutative space considered in [4] is not the quantum Euclidean space and the 'motion' group is not the quantum group $SO_q(N)$.

Everywhere below we suppose that q is not a root of unity. Moreover, under consideration of $*$-operations and scalar products, we suppose that $0 < q < 1$. We shall use two different definitions of q-numbers:

$$[a] = \frac{1-q^a}{1-q} \qquad [a]_q = \frac{q^a - q^{-a}}{q - q^{-1}}.$$

It is necessary to pay attention to which of these definitions is used in each concrete case.

2. The quantum Euclidean space

The quantum Euclidean space E_q^N is defined by means of the algebra of polynomials $\mathcal{A} \equiv \mathbb{C}_q[x_1, x_2, \ldots, x_N]$ in noncommutative elements $x_1, x_2, \ldots, x_N$ which are called quantum Cartesian coordinates; see [5, 6]. The number N can be even or odd and we represent it as $N = 2n$ or $N = 2n + 1$, respectively. Moreover, for $j = 1, 2, \ldots, N$ we shall use the notation $j' = N - j + 1$. The algebra $\mathcal{A}$ is the associative algebra generated by elements $x_1, x_2, \ldots, x_N$ satisfying the defining relations

$$x_i x_j = q x_j x_i \qquad i < j \quad \text{and} \quad i \neq j' \tag{1}$$

$$x_{i'} x_i - x_i x_{i'} = \frac{q - q^{-1}}{q^{\rho_i - 1} + q^{-\rho_i + 1}} \sum_{j=i+1}^{(i+1)'} x_j x_{j'} q^{\rho_{j'}} \qquad i < n \tag{2}$$

$$x_{n'} x_n - x_n x_{n'} = (q^{1/2} - q^{-1/2}) x_{n+1}^2 \qquad \text{if} \quad N = 2n+1 \tag{3}$$

$$x_{n'} x_n = x_n x_{n'} \qquad \text{if} \quad N = 2n \tag{4}$$

where

$$(\rho_1, \ldots, \rho_{2n+1}) = \left(n - \tfrac{1}{2}, n - \tfrac{3}{2}, \ldots, \tfrac{1}{2}, 0, -\tfrac{1}{2}, \ldots, -n + \tfrac{1}{2}\right) \qquad \text{if} \quad N = 2n+1$$

$$(\rho_1, \ldots, \rho_{2n}) = (n-1, n-2, \ldots, 1, 0, 0, -1, \ldots, -n+1) \qquad \text{if} \quad N = 2n.$$

The monomials $\mathbf{x}^\nu := x_1^{\nu_1} x_2^{\nu_2} \cdots x_N^{\nu_N}$, $\nu_i = 0, 1, 2, \ldots$, form a basis of $\mathcal{A}$; see [3]. The vector space of the algebra $\mathcal{A}$ can be represented as a direct sum of the vector subspaces $\mathcal{A}_m$ consisting of homogeneous polynomials of homogeneity degree m: $\mathcal{A} = \bigoplus_{m=0}^{\infty} \mathcal{A}_m$.

A $*$-operation (that is, an involutive algebra anti-automorphism) can be defined on the algebra $\mathcal{A}$ turning it into a $*$-algebra. This $*$-operation is uniquely determined by the relations $x_i^* = q^{\rho_{i'}} x_{i'}$, $i = 1, 2, \ldots, N$.

The quantum rotation group $SO_q(N)$ and the corresponding quantized universal enveloping algebra $U_q(\mathrm{so}_N)$ act on the algebra $\mathcal{A}$. These actions are determined by each other. It will be convenient for us to use the action of the algebra $U_q(\mathrm{so}_N)$. The last algebra is the Hopf algebra generated by the elements K_i, K_i^{-1}, E_i, F_i, $i = 1, 2, \ldots, n$, satisfying the certain defining relations (see, for example, section 6.1.3 of [6]), where n is an integral part of $N/2$. The algebra $U_q(\mathrm{so}_N)$ is supplied by the Hopf algebra operations. We adopt these operations determined in [3]. The action of $X \in U_q(\mathrm{so}_N)$ on an element $a \in \mathcal{A}$ will be denoted as $X \triangleright a$.

A $*$-operation can also be introduced on $U_q(\mathrm{so}_N)$. We adopt such a $*$-operation which defines the compact real form of $U_q(\mathrm{so}_N)$; see, for example, section 6.1.7 of [6]. The action of the $*$-algebra $U_q(\mathrm{so}_N)$ on the $*$-algebra $\mathcal{A}$ is such that $(X \triangleright a)^* = S(X)^* \triangleright a^*$ for $X \in U_q(\mathrm{so}(N))$ and $a \in \mathcal{A}$, where S is the antipode on $U_q(\mathrm{so}_N)$; see [3].

The action of $U_q(\mathrm{so}_N)$ on $\mathcal{A}$ is explicitly given in [3], lemma 2.5. For $U_q(\mathrm{so}_{2n+1})$ and $U_q(\mathrm{so}_{2n})$, the action of elements E_k and F_k, $k = 1, 2, \ldots, n-1$, are determined as

$$E_k \triangleright \mathbf{x}^\nu = [\nu_{k+1}]_q q^{\nu_k - \nu_{k+1}+1} \mathbf{x}^{\nu+\varepsilon_k-\varepsilon_{k+1}} - [\nu_{k'}]_q q^{\nu_k-\nu_{k+1}-\nu_{k'}+\nu_{(k+1)'}+1} \mathbf{x}^{\nu+\varepsilon_{(k+1)'}-\varepsilon_{k'}}$$
$$F_k \triangleright \mathbf{x}^\nu = [\nu_k]_q q^{-\nu_k+\nu_{k+1}-\nu_{(k+1)'}+\nu_{k'}+1} \mathbf{x}^{\nu-\varepsilon_k+\varepsilon_{k+1}} - [\nu_{(k+1)'}]_q q^{-\nu_{(k+1)'}+\nu_{k'}+1} \mathbf{x}^{\nu-\varepsilon_{(k+1)'}+\varepsilon_{k'}}.$$

The action of elements E_n and F_n are given by the formulae

$$E_n \triangleright \mathbf{x}^\nu = [\nu_{n+1}] q^{\nu_n-\nu_{n+1}+3/2} \mathbf{x}^{\nu+\varepsilon_n-\varepsilon_{n+1}} - [\nu_{n+2}]_q q^{\nu_n-\nu_{n+2}+1} \mathbf{x}^{\nu+\varepsilon_{n+1}-\varepsilon_{n+2}}$$
$$F_n \triangleright \mathbf{x}^\nu = [\nu_n]_q q^{-\nu_n+\nu_{n+2}+1/2} \mathbf{x}^{\nu-\varepsilon_n+\varepsilon_{n+1}} - [\nu_{n+1}] q^{-\nu_{n+1}+\nu_{n+2}+1} \mathbf{x}^{\nu-\varepsilon_{n+1}+\varepsilon_{n+2}}$$

if $N = 2n+1$ and by the formulae

$$E_n \triangleright \mathbf{x}^\nu = [\nu_{n+1}]_q q^{\nu_{n-1}-\nu_{n+1}+1} \mathbf{x}^{\nu+\varepsilon_{n-1}-\varepsilon_{n+1}} - [\nu_{n+2}]_q q^{\nu_{n-1}+\nu_n-2\nu_{n+1}-\nu_{n+2}+1} \mathbf{x}^{\nu+\varepsilon_n-\varepsilon_{n+2}}$$
$$F_n \triangleright \mathbf{x}^\nu = [\nu_{n-1}]_q q^{-\nu_{n-1}-2\nu_n+\nu_{n+1}+\nu_{n+2}+1} \mathbf{x}^{\nu-\varepsilon_{n-1}+\varepsilon_{n+1}} - [\nu_n]_q q^{-\nu_n+\nu_{n+2}+1} \mathbf{x}^{\nu-\varepsilon_n+\varepsilon_{n+2}}$$

if $N = 2n$, where ε_i is the vector with the ith coordinate equal to 1 and all others equal to 0.

The monomials $\mathbf{x}^\nu$ are weight vectors with respect to the action of $U_q(\mathrm{so}_N)$ on $\mathcal{A}$. We represent weights λ in the well-known orthogonal coordinate system, that is, as $\lambda = \mu_1\varepsilon_1 + \mu_2\varepsilon_2 + \cdots + \mu_n\varepsilon_n$ (in this system, highest weights are given by the coordinates $\mu_1, \mu_2, \ldots, \mu_n$ such that $\mu_1 \geqslant \mu_2 \geqslant \cdots$). The weight of the monomial $\mathbf{x}^\nu$ is

$$\lambda = (\nu_1 - \nu_{1'})\varepsilon_1 + (\nu_2 - \nu_{2'})\varepsilon_2 + \cdots + (\nu_n - \nu_{n'})\varepsilon_n.$$

The action of the element K_i on the monomial $\mathbf{x}^\nu$ is given by the formula

$$K_i \triangleright \mathbf{x}^\nu = q^{(\nu_i-\nu_{i'})-(\nu_{i+1}-\nu_{(i+1)'})} \mathbf{x}^\nu \qquad i < n$$
$$K_n \triangleright \mathbf{x}^\nu = q^{\nu_n-\nu_{n'}} \mathbf{x}^\nu \qquad \text{if} \quad N = 2n+1$$
$$K_n \triangleright \mathbf{x}^\nu = q^{(\nu_{n-1}-\nu_{(n-1)'})+(\nu_n-\nu_{n'})} \mathbf{x}^\nu \qquad \text{if} \quad N = 2n.$$

It is evident that there exist elements $\hat{K}_i$, $i = 1, 2, \ldots, n$, of the algebra $U_q(\mathrm{so}_N)$ such that

$$\hat{K}_i \triangleright \mathbf{x}^\nu = q^{(\nu_i - \nu_{i'})}\mathbf{x}^\nu. \tag{5}$$

A differential calculus is developed on the quantum Euclidean space which is determined by the R-matrix of the quantum algebra $U_q(\mathrm{so}_N)$. There exist different formulations for this differential calculus. We adopt the definition of the differential operators ∂_i, $i = 1, 2, \ldots, N$, used in [3]. These operators act on the monomials $\mathbf{x}^\nu$ as

$$\partial_k \triangleright \mathbf{x}^\nu = [\nu_k]_q q^{\nu_{k+1}+\cdots+\nu_{1'}}\mathbf{x}^{\nu-\varepsilon_k} \qquad k \leqslant n \tag{6}$$

$$\partial_{n+1} \triangleright \mathbf{x}^\nu = [\nu_{n+1}]q^{\nu_{n'}+\cdots+\nu_{1'}}\mathbf{x}^{\nu-\varepsilon_{n+1}} \qquad \text{if} \quad N = 2n+1$$

$$\partial_{k'} \triangleright \mathbf{x}^\nu = [\nu_{k'}]_q q^{\nu_{k'}+\nu_{(k-1)'}\cdots+\nu_{1'}}\mathbf{x}^{\nu-\varepsilon_{k'}} + \sum_{j=k+1}^{n}[\nu_j]_q[\nu_{j'}]_q(q-q^{-1})q^{\rho_k-\rho_j}q^{d_{kj}}\mathbf{x}^{\nu+\varepsilon_k-\varepsilon_j-\varepsilon_{j'}}$$
$$+ [\nu_{n+1}-1][\nu_{n+1}]\frac{q-q^{-1}}{1+q}q^{\rho_k+2}q^{e_k}\mathbf{x}^{\nu+\varepsilon_k-2\varepsilon_{n+1}} \qquad k \leqslant n \tag{7}$$

where $d_{kj} = \nu_k + \cdots + \nu_{j-1} + \nu_{(j-1)'} + \cdots + \nu_{1'}$ and $e_k = (\nu_k + \cdots + \nu_{1'}) - 2\nu_{n+1}$. The last summand of equation (7) must be omitted for $N = 2n$. The operators ∂_i, $i = 1, 2, \ldots, N$, satisfy the relations

$$\partial_i\partial_j = q^{-1}\partial_j\partial_i \qquad i < j \qquad i \neq j' \tag{8}$$

$$\partial_{i'}\partial_i - \partial_i\partial_{i'} = -\frac{q-q^{-1}}{q^{\rho_i-1}+q^{-\rho_i+1}}\sum_{k=i+1}^{(i+1)'}\partial_k\partial_{k'}q^{\rho_k} \qquad i < n \tag{9}$$

$$\partial_{n'}\partial_n - \partial_n\partial_{n'} = -(q^{1/2}-q^{-1/2})\partial_{n+1}^2 \qquad \text{if} \quad N = 2n+1$$
$$\partial_{n'}\partial_n = \partial_n\partial_{n'} \qquad \text{if} \quad N = 2n. \tag{10}$$

The operators ∂_k and the operators $\hat{x}_i$ of left multiplication by x_i satisfy certain relations which can be represented by means of the quantum R-matrix of the algebra $U_q(\mathrm{so}_N)$. These relations are given in [3]. We need the following:

$$\partial_k\hat{x}_k = \hat{x}_k\partial_k q^{\delta_{kk'}-1} - (q-q^{-1})\sum_{j<k}\hat{x}_j\partial_j + (q-q^{-1})\sigma_k q^{2\rho_{k'}}\hat{x}_{k'}\partial_{k'} + c \tag{11}$$

$$\partial_k\hat{x}_j = \hat{x}_j\partial_k + (q-q^{-1})\sigma_{kj}q^{\rho_{j'}-\rho_k}\hat{x}_{k'}\partial_{j'} \qquad k \neq j,\ j' \tag{12}$$

$$\partial_k\hat{x}_{k'} = q\hat{x}_{k'}\partial_k \qquad k \neq k' \qquad c\hat{x}_k = q\hat{x}_k c \qquad c\partial_k = q^{-1}\partial_k c \tag{13}$$

where $\sigma_k = 1$ if $k > k'$ and $\sigma_k = 0$ otherwise, $\sigma_{kj} = 1$ if $k > j'$ and $\sigma_{kj} = 0$ otherwise, c is the linear operator which acts on the monomials $\mathbf{x}^\nu$ as $c \triangleright \mathbf{x}^\nu = q^{\nu_1+\cdots+\nu_{1'}}\mathbf{x}^\nu$.

3. Squared q-radius and q-Laplace operator

The element

$$Q = \sum_{i=1}^{N} q^{\rho_{i'}}x_i x_{i'} = (1+q^{N-2})\left(\sum_{i=1}^{n} q^{\rho_{i'}}x_i x_{i'} + \frac{q}{q+1}x_{n+1}^2\right) \tag{14}$$

(where the last summand with x_{n+1} must be omitted if $N = 2n$) of the algebra $\mathcal{A}$ is called the *squared q-radius* on the quantum Euclidean space. It is shown in [3] that the centre of $\mathcal{A}$ is generated by Q.

We shall also use the elements $Q_j = \sum_{i=j}^{j'} q^{\rho_{i'}} x_i x_{i'}$, $1 < j \leqslant n$, which are squared q-radii for the subalgebras $\mathbb{C}_q[x_j, \ldots, x_{j'}]$. They satisfy the relations

$$Q_j Q_k = Q_k Q_j \qquad x_i x_{i'} = q^{\rho_i}\left(\frac{Q_i}{1+q^{N-2i}} - \frac{Q_{i+1}}{1+q^{N-2i-2}}\right) \qquad 1 \leqslant i \leqslant n$$

$$x_i Q_j = q^2 Q_j x_i \qquad x_{i'} Q_j = q^{-2} Q_j x_{i'} \qquad \text{for} \quad i < j$$

$$x_i Q_j = Q_j x_i \qquad \text{for} \quad j \leqslant i \leqslant j'.$$

It can be checked by direct computation that

$$x_1^k x_{1'}^k = Q'^k_1 (Q'_2/Q'_1; q^2)_k \tag{15}$$

where $Q'_1 = q^{-\rho_{1'}} Q_1/(1+q^{N-2})$, $Q'_2 = q^{-\rho_{1'}} Q_2/(1+q^{N-4})$ and

$$(a;q)_s = (1-a)(1-aq)\cdots(1-aq^{s-1}).$$

To the element Q there corresponds the operator $\hat{Q}$ on $\mathcal{A}$ defined as $\hat{Q} = \sum_{i=1}^{N} q^{\rho_{i'}} \hat{x}_i \hat{x}_{i'}$, where $\hat{x}_i$ is the operator of left multiplication by x_i. It is clear that $\hat{Q} : \mathcal{A}_m \to \mathcal{A}_{m+2}$.

We also consider on $\mathcal{A}$ the operator

$$\Delta_q = \sum_{i=1}^{N} q^{\rho_i} \partial_i \partial_{i'} \tag{16}$$

which is called the *q-Laplace operator* on the quantum Euclidean space. We have $\Delta_q : \mathcal{A}_m \to \mathcal{A}_{m-2}$. The important property of the operators $\hat{Q}$ and Δ_q is that they commute with the action of the algebra $U_q(\mathrm{so}_N)$ on $\mathcal{A}$; see [3]. The operators $\hat{Q}$ and Δ_q satisfy the relations

$$\Delta_q \hat{Q}^k - q^{2k} \hat{Q}^k \Delta_q = \hat{Q}^{k-1} q^{-N+3} [2k][N+2k+2\gamma-2] \frac{(1+q^{N-2})^2}{(1+q)^2} \tag{17}$$

where γ is the operator acting on the monomials $\mathbf{x}^\nu$ as $\gamma \mathbf{x}^\nu = (\nu_1 + \cdots + \nu_N)\mathbf{x}^\nu$ (see [3]). We shall also use the following formula from [3]

$$\Delta_q(\mathbf{x}^\nu) = (1+q^{N-2}) q^{\nu_1+\cdots+\nu_{1'}-1} \times \left(\sum_{j=1}^{n} [\nu_j]_q [\nu_{j'}]_q q^{-\rho_j} q^d \mathbf{x}^{\nu-\varepsilon_j-\varepsilon_{j'}} + [\nu_{n+1}-1][\nu_{n+1}] \frac{q^e}{1+q} \mathbf{x}^{\nu-2\varepsilon_{n+1}}\right) \tag{18}$$

where $d = \nu_1 + \cdots + \nu_{j-1} + \nu_{(j-1)'} + \cdots + \nu_{1'}$, $e = \nu_1 + \cdots + \nu_{1'} - 2\nu_{n+1} + 2$, and the last summand must be omitted for $N = 2n$.

4. *q*-Harmonic polynomials

A polynomial $p \in \mathcal{A}$ is called *q-harmonic* if $\Delta_q p = 0$. The linear subspace of $\mathcal{A}$ consisting of all q-harmonic polynomials is denoted by $\mathcal{H}$. If $\mathcal{H}_m = \mathcal{A}_m \cap \mathcal{H}$, then $\mathcal{H}_m$ is the subspace of $\mathcal{H}$ consisting of all homogeneous of degree m harmonic polynomials.

Remark. If $n = 2$, then $\mathcal{A}$ consists of all polynomials in commuting elements x_1 and $x_{1'} \equiv x_2$. In this case, the space $\mathcal{H}$ of q-harmonic polynomials has a basis consisting of the polynomials

$$1, x_1^k, x_{1'}^k \qquad k = 1, 2, \ldots. \tag{19}$$

Proposition 1 [3]. *If $m \geqslant 2$, then the space $\mathcal{A}_m$ can be represented as the direct sum*

$$\mathcal{A}_m = \mathcal{H}_m \oplus Q\mathcal{A}_{m-2}. \tag{20}$$

We shall need the following consequences of the decomposition (20):

Corollary 1. *If $p \in \mathcal{H}_m$, then p cannot be represented as $p = Q^k p'$, $k \neq 0$, with some polynomial $p' \in \mathcal{A}$.*

Corollary 2. *For dimension of the space of q-harmonic polynomials $\mathcal{H}_m$ we have the formula*

$$\dim \mathcal{H}_m = \frac{(m+N-3)!(2m+N-2)}{(N-2)!m!}.$$

Corollary 3. *The linear space $\mathcal{H}$ can be represented as a direct sum $\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$.*

Corollary 1 is a direct consequence of formula (20). Corollary 2 is proven in the same way as in the classical case; see, for example, chapter 10 of [1]. For this we note that $\dim \mathcal{A}_m = \frac{(N+m-1)!}{(N-1)!m!}$. Hence, for $\dim \mathcal{H}_m = \dim \mathcal{A}_m - \dim \mathcal{A}_{m-2}$ we obtain the expression stated in the corollary. In order to prove corollary 3, we note that any $p \in \mathcal{H}$ can be represented as $p = \sum_m p_m$, $p_m \in \mathcal{A}_m$. We have $\Delta_q p = \sum_m \Delta_q p_m = 0$. Since $\Delta_q p_m$, $m = 0, 1, 2, \ldots,$ have different homogeneity degrees, it follows from the last equality that $\Delta_q p_m = 0$ for all values of m. Thus, $\mathcal{H} = \bigoplus_{m=0}^{\infty} \mathcal{H}_m$.

Proposition 2. *The linear space isomorphism $\mathcal{A} \simeq \mathbb{C}[Q] \otimes \mathcal{H}$ is true, where $\mathbb{C}[Q]$ is the space of all polynomials in Q.*

This proposition follows from the fact that the space $\mathcal{A}_m$ decomposes into the direct sum $\mathcal{A}_m = \bigoplus_{j=0}^{\lfloor m/2 \rfloor} Q^j \mathcal{H}_{m-2j}$, where $\lfloor m/2 \rfloor$ is the integral part of the number $m/2$.

The decomposition $\mathcal{A} \simeq \mathbb{C}[Q] \otimes \mathcal{H}$ is a q-analogue of the theorem on separation of variables for Lie groups in an abstract form [7]. Thus, a study of $\mathcal{A}$ reduces to a study of the space $\mathcal{H}$.

Since the operator Δ_q commutes with the action of the algebra $U_q(\mathrm{so}_N)$, the subspaces $\mathcal{H}_m$ are invariant with respect to the action of this algebra. It is proven in [3] that the irreducible representation T_m of $U_q(\mathrm{so}_N)$ with highest weight $(m, 0, \ldots, 0)$ is realized on $\mathcal{H}_m$.

We denote by $\mathcal{A}^{U_q(\mathrm{so}_r)}$ the space of elements of $\mathcal{A}$ consisting of invariant elements with respect to the action of $U_q(\mathrm{so}_r) \subset U_q(\mathrm{so}_N)$. We have $\mathcal{A}^{U_q(\mathrm{so}_N)} = \mathbb{C}[Q]$; see [3]. In what follows we shall consider the subalgebra $U_q(\mathrm{so}_{N-2})$ generated by the elements H_i, E_i, F_i, $i = 2, 3, \ldots, n$.

Proposition 3. *We have $\mathcal{A}^{U_q(\mathrm{so}_{N-2})} \simeq \bigoplus_{k,l} \mathbb{C}[Q_2] x_1^k x_{1'}^l \simeq \bigoplus_{k,l} \mathbb{C}[Q] x_1^k x_{1'}^l$.*

Proof. In order to prove this proposition we note that for $U_q(\mathrm{so}_{N-2})$-module $\mathcal{A}$ we have

$$\mathcal{A} = \mathbb{C}_q[x_1, x_2, \ldots, x_N] = \bigoplus_{k,l} \mathbb{C}_q[x_2, x_3, \ldots, x_{N-1}] x_1^k x_N^l.$$

The action of $U_q(\mathrm{so}_{N-2})$ on $x_1^k x_N^l$ is trivial. Moreover, $\mathbb{C}_q[x_2, x_3, \ldots, x_{N-1}]^{U_q(\mathrm{so}_{N-2})} = \mathbb{C}[Q_2]$. Since $Q = c_1 Q_2 + c_2 x_1 x_N$, where c_1 and c_2 are constants, we have $\mathcal{A}^{U_q(\mathrm{so}_{N-2})} \simeq \bigoplus_{k,l} \mathbb{C}[Q_2] x_1^k x_N^l \simeq \bigoplus_{k,l} \mathbb{C}[Q] x_1^k x_N^l$. Proposition is proven. □

The associative algebra $\mathcal{F}(S_q^{N-1})$ generated by the elements $x_1, \ldots, x_N$ satisfying the relations (1)–(3) and the relation $Q = 1$ is called *the algebra of functions on the quantum sphere* S_q^{N-1}; see [5] and chapter 11 of [6]. It is clear that the canonical algebra isomorphism $\mathcal{F}(S_q^{N-1}) \simeq \mathcal{A}/\mathcal{I}$ is true, where $\mathcal{I}$ is the two-sided ideal of $\mathcal{A}$ generated by the element $Q - 1$. We denote by τ the canonical algebra homomorphism $\tau : \mathcal{A} \to \mathcal{A}/\mathcal{I} \simeq \mathcal{F}(S_q^{N-1})$. This homomorphism is called the *restriction* of polynomials of $\mathcal{A}$ on to the quantum sphere S_q^{N-1}.

It was shown in [3] that $\tau : \mathcal{H} \to \mathcal{F}\big(S_q^{N-1}\big)$ is a one-to-one mapping, that is, the restriction of a q-harmonic polynomial to the sphere S_q^{N-1} determines this polynomial uniquely. This statement allows us to determine a scalar product on $\mathcal{H}$. For this, we use the invariant functional h on the quantum sphere S_q^{N-1} defined in [3]. In order to give this functional, we introduce the linear subspace $(\tau\mathcal{A})^0$ of $\mathcal{F}\big(S_q^{N-1}\big)$ spanned by the elements $\tau\mathbf{x}^\nu$ such that $\nu_1 = \nu_{1'}, \ldots, \nu_n = \nu_{n'}, \nu_{n+1} = 2m$, $m = 0, 1, 2, \ldots$ (for $N = 2n$ the last condition must be omitted). The functional h vanishes on the elements $\tau\mathbf{x}^\nu \notin (\tau\mathcal{A})^0$ and on the monomials $\tau\mathbf{x}^\nu \in (\tau\mathcal{A})^0$ it is given by formula (5.2) of [3]. A scalar product $\langle\cdot\,,\cdot\rangle$ on $\mathcal{H}$ is introduced by the formula $\langle p_1, p_2\rangle = h((\tau p_1)^*(\tau p_2))$, where a^* determines an element conjugate to $a \in \mathcal{A}$ under action of the $*$-operation.

Proposition 4. *We have $\mathcal{H}_m \perp \mathcal{H}_r$ if $m \neq r$.*

Proof follows from the fact that $(\tau p_1)^*(\tau p_2) \notin (\tau\mathcal{A})^0$ if $p_1 \in \mathcal{H}_m$, $p_2 \in \mathcal{H}_r$, $m \neq r$.

Note that the operators

$$\omega(k) = q^{N/2}q^{\gamma} \qquad \omega(e) = \hat{Q} \qquad \omega(f) = -\Delta_q q^{-\gamma} q^{N/2}/(1+q^{N-2})^2$$

satisfy the relations $ke = q^2ek$, $kf = q^{-2}fk$, $ef - fe = (k-k^{-1})/(q-q^{-1})$, determining the quantum algebra $U_q(\mathrm{sl}_2)$. Therefore, the algebra homomorphism $\omega : U_q(\mathrm{sl}_2) \to \mathcal{L}(\mathcal{A})$ gives a representation of $U_q(\mathrm{sl}_2)$, commuting with the natural action L of $U_q(\mathrm{so}_N)$ on $\mathcal{A}$ determined above. It is shown as in [2] that the representation $\omega \times L$ of $U_q(\mathrm{sl}_2) \times U_q(\mathrm{so}_N)$ decomposes into irreducible representations as

$$\omega \times L = \bigoplus_{m=0}^{\infty} D_{m+N/2} \times T_m$$

where $D_{m+N/2}$ are the discrete series representations of $U_q(\mathrm{sl}_2)$ with lowest weights $m + N/2$; see, for example, [8]. Therefore, $U_q(\mathrm{sl}_2)$ and $U_q(\mathrm{so}_N)$ constitute a quantum dual pair under the action on $\mathcal{A}$. Note that this dual pair is different from that of [4] since the algebra $U_q(\mathrm{so}_N)$ of [4] is not a Drinfeld–Jimbo algebra.

5. The projection $\mathcal{A}_m \to \mathcal{H}_m$

Let us go back to the decomposition (20) and construct the projector $\mathsf{H}_m : \mathcal{A}_m = \mathcal{H}_m \oplus Q\mathcal{A}_{m-2} \to \mathcal{H}_m$. We present this projector in the form

$$\mathsf{H}_m = \sum_{k=0}^{\lfloor m/2\rfloor} \alpha_k \hat{Q}^k \Delta_q^k \qquad \alpha_k \in \mathbb{C} \tag{21}$$

where $\lfloor m/2\rfloor$ is the integral part of the number $m/2$. Let us show that the summands on the right-hand side are linearly independent (in this case, the coefficients α_k are determined uniquely up to a common constant). Let $p = x_{n+1}^m$ if $N = 2n + 1$. Using formula (18) we derive that

$$\Delta_q^k\left(x_{n+1}^m\right) = q^k\left(\frac{1+q^{N-2}}{1+q}\right)^k \frac{[m]!}{[m-2k]!} x_{n+1}^{m-2k} \qquad 2k \leqslant m \tag{22}$$

where $[m]! = [1][2]\cdots[m]$. Acting by the right-hand side of (21) on x_{n+1}^m, we obtain a linear combination of the elements $Q^k x_{n+1}^{m-2k}$, $k = 1, 2, \ldots, \lfloor m/2\rfloor$. It is easy to see that these elements are linearly independent. This means that the summands in equation (21) are linearly independent. If $N = 2n$, then instead of $p = x_{n+1}^m$ we take $p = x_1^{m_1}x_{1'}^{m_1'}$ and make the same reasoning (see this calculation in the next section).

We have to calculate values of the coefficients α_k in equation (21). In order to do this we take the relation $\mathsf{H}_m p = \sum_k \alpha_k \hat{Q}^k \Delta_q^k p$, $p \in \mathcal{A}_m$, act by the operator Δ_q upon both sides of it and use the relation (17). Under this action, the left-hand side vanishes. Equating the right-hand side to 0 and taking into account that the elements $\hat{Q}^k \Delta_q^{k+1} p$, $k = 1, 2, \ldots, \lfloor m/2 \rfloor$, are linearly independent for generic elements $p \in \mathcal{A}_m$, we derive the recurrence relation

$$q^{-N-2k+5}(1+q^{N-2})^2[2k][N+2m-2k-2]\alpha_k + (1+q)^2\alpha_{k-1} = 0$$

for α_k. This gives

$$\alpha_k = (-1)^k q^{(N-4)k+k^2} \frac{(1+q)^{2k}}{(1+q^{N-2})^{2k}} \frac{[N+2m-2k-4]!!}{[2k]!![N+2m-4]!!}$$

where $[s]!! = [s][s-2][s-4]\cdots[2](\text{or}[1])$ for $s \neq 0$ and $[0]!! = 1$. Using the relations

$$[2k]!! = \frac{(q^2;q^2)_k}{(1-q)^k} \qquad \frac{[N+2m-2k-4]!!}{[N+2m-4]!!} = \frac{(1-q)^k}{(q^{N+2m-2k-2};q^2)_k}$$

$$(q^{N+2m-2k-2};q^2)_k = (q^{-N-2m+4};q^2)_k q^{-2k-k(k-1)}(-q^{N+2m-2})^k$$

we derive that

$$\alpha_k = \frac{q^{2k^2-2mk-k}(1-q^2)^{2k}}{(1+q^{N-2})^{2k}(q^{-N-2m+4};q^2)_k(q^2;q^2)_k}. \tag{23}$$

Note that the coefficients α_k are determined by the recurrence relation uniquely up to a common constant. In equation (21) we have chosen this constant in such a way that $\mathsf{H}_m p = p$ for $p \in \mathcal{H}_m$. This means that $\mathsf{H}_m^2 = \mathsf{H}_m$.

Since the action of $X \in U_q(\mathrm{so}_N)$ commutes with $\hat{Q}$ and Δ_q the operator H_m commutes with the action of $U_q(\mathrm{so}_N)$.

The operator H_m can be used for obtaining explicit forms of q-harmonic polynomials. As an example, we derive here formulae for harmonic projection of the polynomial $x_{n+1}^m \in \mathcal{A}_m$ when $N = 2n+1$. For this, we use formula (22) for $\Delta_q^k(x_{n+1}^m)$. Since

$$\frac{[m]!}{[m-2k]!} = \frac{(q^{m-2k+2};q^2)_k(q^{m-2k+1};q^2)_k}{(1-q)^{-2k}}$$

$$(q^{m-2k+2};q^2)_k = (-1)^k q^{mk-k(k-1)}(q^{-m};q^2)_k$$

$$(q^{m-2k+1};q^2)_k = (-1)^k q^{mk-k^2}(q^{-m+1};q^2)_k$$

we derive that

$$\Delta_q^k\left(x_{n+1}^m\right) = \left(\frac{1+q^{N-2}}{1+q}\right)^k \frac{q^{2mk-2k^2+2k}}{(1-q)^{2k}}(q^{-m};q^2)_k(q^{-m+1};q^2)_k x_{n+1}^{m-2k}.$$

Using expression (21) for $\mathsf{H}_m x_{n+1}^m$ and formula (23) for coefficients α_k, we obtain

$$\mathsf{H}_m x_{n+1}^m = x_{n+1}^m \sum_{k=0}^{\lfloor m/2 \rfloor} \frac{(q^{-m};q^2)_k(q^{-m+1};q^2)_k}{(q^2;q^2)_k(q^{-N-2m+4};q^2)_k}\left(aQx_{n+1}^{-2}\right)^k \qquad a = \frac{q(1+q)}{1+q^{N-2}}. \tag{24}$$

Note that we have used x_{n+1}^{-2} in equation (24). However, since there exists the multiplier x_{n+1}^m before the sum sign, negative powers of x_{n+1} in fact are absent.

Expression (24) for $\mathsf{H}_m x_{n+1}^m$ can be represented in terms of the basic hypergeometric function ${}_2\phi_1$ (see [9] for the definition of this function):

$$\mathsf{H}_m x_{n+1}^m = x_{n+1}^m {}_2\phi_1\left(q^{-m}, q^{-m+1}; q^{-N-2m+4}; q^2, aQx_{n+1}^{-2}\right).$$

Using the definition $P_k^{(\alpha,\beta)}(x;q) = {}_2\phi_1(q^{-k}, q^{\alpha+\beta+k+1}; q^{\alpha+1}; q, qx)$ of the little q-Jacobi polynomials, we can represent $\mathsf{H}_m x_{n+1}^m$ in the form

$$\mathsf{H}_m x_{n+1}^m = x_{n+1}^m P_r^{(-\frac{N}{2}+m+1,\frac{N-3}{2})}\left(\frac{1+q}{q(1+q^{N-2})}Qx_{n+1}^{-2}\right)$$

where $r = m/2$ if m is even and $r = (m-1)/2$ if m is odd.

6. Zonal polynomials with respect to $SO_q(N-2)$

A polynomial φ of the space $\mathcal{H}_m$ is called *zonal* with respect to the quantum subgroup $SO_q(N-2)$ (or with respect to the subalgebra $U_q(\mathrm{so}_{N-2})$) if it is invariant with respect to the action of elements $X \in U_q(\mathrm{so}_{N-2})$. In order to find zonal polynomials $\varphi \in \mathcal{H}_m$ we have to take polynomials $p \in \mathcal{A}_m$ invariant with respect to the subalgebra $U_q(\mathrm{so}_{N-2})$ and to act on them by the projection H_m.

It follows from proposition 3 that, in the space $\mathcal{A}_m$, there exist $m+1$ elements which are $U_q(\mathrm{so}_{N-2})$-invariant and linearly independent over $\mathbb{C}[Q]$. They coincide with $x_1^{m_1}x_{1'}^{m_1'}$, $m_1+m_1' = m$. Therefore, $\mathsf{H}_m\left(x_1^{m_1}x_{1'}^{m_1'}\right)$, $m_1+m_1' = m$, are zonal polynomials with respect to $U_q(\mathrm{so}_{N-2})$. Let us find an explicit form of these polynomials.

Using formula (18) we find that

$$\Delta_q^k\left(x_1^{m_1}x_{1'}^{m_1'}\right) = (1+q^{N-2})^k q^{(m-k)k} q^{-(n-\epsilon)k}\frac{[m_1]_q![m_1']_q!}{[m_1-k]_q![m_1'-k]_q!}x_1^{m_1-k}x_{1'}^{m_1'-k}$$

where $\epsilon = 1$ for $N = 2n$ and $\epsilon = \frac{1}{2}$ for $N = 2n+1$. Since

$$\frac{[m_1]_q!}{[m_1-k]_q!} = q^{(2m_1-k+1)k/2}\frac{(q^{-2m_1};q^2)_k}{(q-q^{-1})^k}$$

we have

$$\Delta_q^k\left(x_1^{m_1}x_{1'}^{m_1'}\right) = (1+q^{N-2})^k q^{2(m-k)k} q^{(-n+3+\epsilon)k}\frac{(q^{-2m_1};q^2)_k(q^{-2m_1'};q^2)_k}{(1-q^2)^{2k}}x_1^{m_1-k}x_{1'}^{m_1'-k}.$$

Now using formulae (21) and (23) we derive that

$$\varphi_{m_1m_1'}^m \equiv \mathsf{H}_m\left(x_1^{m_1}x_{1'}^{m_1'}\right) = \sum_{k=0}^{\min(m_1,m_1')}\frac{(q^{-2m_1};q^2)_k(q^{-2m_1'};q^2)_k}{(q^2;q^2)_k(q^{-N-2m+4};q^2)_k}\frac{q^{(-n+2+\epsilon)k}}{(1+q^{N-2})^k}Q^k x_1^{m_1-k}x_{1'}^{m_1'-k}. \tag{25}$$

Using formula (15) the polynomials $\varphi_{m_1m_1'}^m$ can be represented as

$$\varphi_{m_1m_1'}^m = x_1^{m_1-m_1'}\sum_{k=0}^{m_1'}C_{m_1m_1'}^k Q^k Q'^k (Q_2'/Q';q^2)_{m'-k} \tag{26}$$

if $m_1 \geqslant m_1'$ and as

$$\varphi_{m_1m_1'}^m = \left(\sum_{k=0}^{m_1}C_{m_1m_1'}^k Q^k Q'^k (Q_2'/Q';q^2)_{m'-k}\right)x_{1'}^{m_1'-m_1} \tag{27}$$

if $m_1' \geqslant m_1$, where $Q' \equiv Q_1'$ and Q_2' are such as in equation (15). Unfortunately, these polynomials cannot be represented in terms of known orthogonal polynomials, as it was in the case of the quantum spaces on which the quantum group $GL_q(N)$ acts; see [2].

Theorem 1. *The zonal polynomials $\varphi^m_{m_1 m_1'}$, $m_1 + m_1' = m$, of $\mathcal{H}_m$ are orthogonal with respect to the scalar product introduced in section 4. These polynomials constitute a full set of zonal polynomials in the space $\mathcal{H}_m$.*

Proof. We have $\hat{K}_1 \triangleright \left(x_1^{m_1} x_{1'}^{m_1'}\right) = q^{m_1 - m_1'} \left(x_1^{m_1} x_{1'}^{m_1'}\right)$; that is, the monomials $x_1^{m_1} x_{1'}^{m_1'}$, $m_1 + m_1' = m$, are eigenfunctions of the operator defined by the action of $\hat{K}_1$ on $\mathcal{A}$ which belong to different eigenvalues. Since the projection $\mathsf{H}_m : \mathcal{A}_m \to \mathcal{H}_m$ commutes with the action of $U_q(\mathrm{so}_N)$, then $\hat{K}_1 \triangleright \varphi^m_{m_1 m_1'} = q^{m_1 - m_1'} \varphi^m_{m_1 m_1'}$. The scalar product of section 4 is defined in terms of the invariant functional; that is, this scalar product is invariant with respect to the action of $\hat{K}_i$, $i = 1, 2, \ldots, n$. Since the zonal polynomials $\varphi^m_{m_1 m_1'}$, $m_1 + m_1' = m$, belong to different eigenvalues of $\hat{K}_1$, they are orthogonal. The theorem is proven. □

It is possible to define zonal polynomials of the space $\mathcal{H}_m$ with respect to the subalgebra $A := U_q(\mathrm{so}_2) \times U_q(\mathrm{so}_{N-2})$, where $U_q(\mathrm{so}_2)$ is the subalgebra of $U_q(\mathrm{so}_N)$ generated by the element $\hat{K}_1$. Then the following assertions are true which easily follows from the above results.

Theorem 2. *The subspace of zonal polynomials of the space $\mathcal{H}_m$ with respect to the subalgebra A is not more than one-dimensional. The space $\mathcal{H}_m$ contains a zonal polynomial if and only if m is even. This zonal polynomial coincides with the polynomial $\varphi^m_{m/2,m/2}$ given by formula (25).*

7. Associated spherical polynomials with respect to $SO_q(N-2)$

The aim of this section is to construct an orthogonal basis of the space $\mathcal{H}_m$ of homogeneous q-harmonic polynomials which corresponds to the chain $U_q(\mathrm{so}_N) \supset U_q(\mathrm{so}_{N-2}) \supset \cdots \supset U_q(\mathrm{so}_3)$ (or $U_q(\mathrm{so}_2)$). This basis is a q-analogue of the set of associated spherical harmonics on the classical Euclidean space which are products of Jacobi polynomials and correspond to the chain of the subgroups $SO(N) \supset SO(2) \times SO(N-2) \supset \cdots$; see chapter 10 of [1]. The basis elements give solutions of the equation $\Delta_q p = 0$ in 'separated coordinates'. So, we obtain a q-analogue of the classical separation of variables.

Let us note that

$$\Delta_q \equiv \Delta_q^{(N)} = \sum_{j=1}^{N} q^{\rho_j} \partial_j \partial_{j'} = (q^{\rho_1} \partial_1 \partial_{1'} + q^{\rho_{1'}} \partial_{1'} \partial_1) + \Delta_q^{(N-2)} \tag{28}$$

where $\Delta_q^{(N-2)}$ is the q-Laplace operator on the subspace $\mathcal{A}^{(N-2)} \equiv \mathbb{C}_q[x_2, \ldots, x_{2'}]$. We also have from equation (9) that

$$\partial_{1'} \partial_1 - \partial_1 \partial_{1'} = -\frac{q - q^{-1}}{q^{\rho_1 - 1} + q^{-\rho_1 + 1}} \Delta_q^{(N-2)}. \tag{29}$$

Let $p(x_2, \ldots, x_{2'})$ be a polynomial of $\mathcal{A}$ which does not depend on x_1 and $x_{1'} \equiv x_N$. Then it is easy to see from equation (6) that $\partial_1 p(x_2, \ldots, x_{2'}) = 0$.

Lemma 1. *Let $p(x_2, \ldots, x_{2'}) \in \mathcal{A}$ and $\Delta_q^{(N-2)} p = 0$. Then $\partial_{1'} p = 0$ and $\Delta_q p = 0$.*

Proof. Let $p(x_2, \ldots, x_{2'})$ be harmonic with respect to $\Delta_q^{(N-2)}$, that is $\Delta_q^{(N-2)} p = 0$. Then due to equation (29) we have $\partial_1 \partial_{1'} p = 0$, and $\Delta_q p = 0$ using equation (28). From formula (7) for $\partial_{1'}$, it follows that $\partial_{1'} p = x_1 p'(x_2, \ldots, x_{2'})$, where $p'(x_2, \ldots, x_{2'})$ is a polynomial in $x_2, x_3, \ldots, x_{2'}$. Let us show that from $\partial_1 \partial_{1'} p = 0$ the equality $\partial_{1'} p = 0$

follows. Indeed, due to equation (6) we have $0 = \partial_1\partial_{1'}p = \partial_1 x_1 p' = \tilde{p}(x_2, \ldots, x_{2'})$, where $\tilde{p}(x_2, \ldots, x_{2'})$ is some polynomial in $x_2, x_3, \ldots, x_{2'}$ which is a linear combination (with nonvanishing coefficients) of the same monomials as p' has. Moreover, if $p' \neq 0$ then $\tilde{p} \neq 0$. Since $\tilde{p} = 0$ then $p' = 0$ and we have $\partial_{1'}p = x_1 p'(x_2, \ldots, x_{2'}) = 0$. This proves the lemma. □

Lemma 2. *If $p(x_2, \ldots, x_{2'}) \in \mathcal{A}$, then*

$$\Delta_q^{(N-2)}\big(x_1^{m_1}x_{1'}^{m_1'}p(x_2, \ldots, x_{2'})\big) = x_1^{m_1}x_{1'}^{m_1'}\Delta_q^{(N-2)}p(x_2, \ldots, x_{2'}). \tag{30}$$

If $p(x_2, \ldots, x_{2'})$ is $\Delta_q^{(N-2)}$-harmonic, then

$$\Delta_q\big(x_1^{m_1}x_{1'}^{m_1'}p(x_2, \ldots, x_{2'})\big) = (q^{\rho_1} + q^{-\rho_1})\partial_{1'}\partial_1 x_1^{m_1}x_{1'}^{m_1'}p(x_2, \ldots, x_{2'}). \tag{31}$$

Proof. Since $\partial_1\hat{x}_{1'} = q\hat{x}_{1'}\partial_1$, $\partial_2\hat{x}_1 = \hat{x}_1\partial_2$, $\partial_1 p(x_2, \ldots, x_{2'}) = 0$, and $\partial_2\hat{x}_{1'} = \hat{x}_{1'}\partial_2 + (q - q^{-1})q^{\rho_1-\rho_2}\hat{x}_{2'}\partial_1$ (see formulae (11)–(13)) we obtain

$$\begin{aligned}\partial_2\big(x_1^{m_1}x_{1'}^{m_1'}p(x_2, \ldots, x_{2'})\big) &= \hat{x}_1^{m_1}\partial_2\big(x_{1'}^{m_1'}p(x_2, \ldots, x_{2'})\big)\\ &= \hat{x}_1^{m_1}(\hat{x}_{1'}\partial_2 + (q - q^{-1})q^{\rho_1-\rho_2}\hat{x}_{2'}\partial_1)x_{1'}^{m_1'-1}p(x_2, \ldots, x_{2'})\\ &= \hat{x}_1^{m_1}\hat{x}_{1'}^{m_1'}\partial_2 p(x_2, \ldots, x_{2'}).\end{aligned}$$

Analogously, using relation (12) we derive that

$$\begin{aligned}\partial_{2'}\big(x_1^{m_1}x_{1'}^{m_1'}p(x_2, \ldots, x_{2'})\big) &= \hat{x}_1^{m_1}\partial_{2'}\big(x_{1'}^{m_1'}p(x_2, \ldots, x_{2'})\big)\\ &= \hat{x}_1^{m_1}(\hat{x}_{1'}\partial_{2'} + (q - q^{-1})q^{\rho_1-\rho_{2'}}\hat{x}_2\partial_1)\big(x_{1'}^{m_1'-1}p(x_2, \ldots, x_{2'})\big)\\ &= \hat{x}_1^{m_1}\hat{x}_{1'}^{m_1'}\partial_{2'}p(x_2, \ldots, x_{2'}).\end{aligned}$$

We have the same results when ∂_2 and $\partial_{2'}$ are replaced by ∂_i and $\partial_{i'}$, $i = 3, 4, \ldots$. This leads to the relation (30). If p is $\Delta_q^{(N-2)}$-harmonic, then it follows from equation (29) that $(\partial_{1'}\partial_1 - \partial_1\partial_{1'})x_1^{m_1}x_{1'}^{m_1'}p(x_2, \ldots, x_{2'}) = 0$. From here and from equation (28) we derive that

$$\Delta_q\big(x_1^{m_1}x_{1'}^{m_1'}p\big) = (q^{\rho_1}\partial_1\partial_{1'} + q^{\rho_{1'}}\partial_{1'}\partial_1)\big(x_1^{m_1}x_{1'}^{m_1'}p\big) = (q^{\rho_1} + q^{\rho_{1'}})\partial_{1'}\partial_1\big(x_1^{m_1}x_{1'}^{m_1'}p\big)$$

and the relation (31) is proven. The lemma is proven. □

Proposition 5. *Let $h \equiv h(x_2, \ldots, x_{2'})$ be a $\Delta_q^{(N-2)}$-harmonic polynomial of degree l. Then*

$$\Delta_q\big(x_1^{m_1}x_{1'}^{m_1'}h\big) = (q^{\rho_1} + q^{-\rho_1})[m_1]_q[m_1']_q q^{m_1+m_1'-1}x_1^{m_1-1}x_{1'}^{m_1'-1}h. \tag{32}$$

Proof. Using equation (31) and then equations (1), (6) and (13), we derive that

$$\Delta_q\big(x_1^{m_1}x_{1'}^{m_1'}h(x_2, \ldots, x_{2'})\big) = (q^{\rho_1} + q^{-\rho_1})[m_1]_q q^{m_1'+m_1+l-1}x_1^{m_1-1}\partial_{1'}x_{1'}^{m_1'}h(x_2, \ldots, x_{2'}). \tag{33}$$

Using equation (11) we have for $\partial_{1'}x_{1'}^{m_1'}h(x_2, \ldots, x_{2'})$ the expression

$$\begin{aligned}\partial_{1'}x_{1'}^{m_1'}h &= \left(q^{-1}x_{1'}\partial_{1'} - (q - q^{-1})\sum_{j<N}\hat{x}_j\partial_j + (q - q^{-1})\hat{x}_1\partial_1 q^{2\rho_1} + c\right)x_{1'}^{m_1'-1}h\\ &= (q\hat{x}_{1'}\partial_{1'} - (q - q^{-1})E + q^{m_1'-1+l})x_{1'}^{m_1'-1}h\end{aligned} \tag{34}$$

where $E = \sum_{k=1}^{N}\hat{x}_k\partial_k$. It is proven by using the relation between $\hat{x}_i$ and ∂_j that

$$E\hat{x}_k = q^{-1}\hat{x}_k E + \frac{q - q^{-1}}{1 + q^{N-2}}q^{N-\rho_k-2}\hat{Q}\partial_{k'} + \hat{x}_k c$$

(see also proposition 2.9 of [3]). Then

$$E\big(x_{1'}^{m_1'}h(x_2,\ldots,x_{2'})\big) = (q^{-1}\hat{x}_{1'}E + x_{1'}q^{l+m_1'-1})x_{1'}^{m_1'-1}h(x_2,\ldots,x_{2'})$$
$$= (q^{l+m_1'-1} + q^{l+m_1'-3} + \cdots + q^{l-m_1'+1})x_{1'}^{m_1'}$$
$$\times h(x_2,\ldots,x_{2'}) + q^{-m_1'}\hat{x}_{1'}^{m_1'}Eh(x_2,\ldots,x_{2'}).$$

By direct calculation it is proven that

$$E = \frac{c-c^{-1}}{q-q^{-1}} + \frac{q-q^{-1}}{(1+q^{N-2})^2}q^{N-1}\hat{Q}\Delta_q c^{-1}$$

(see also [3]). Since $h \in \mathcal{H}_l$, then $Eh = [l]_q h$. Now we have for $E\hat{x}_{1'}^{m_1'}h$ the expression

$$E\big(x_{1'}^{m_1'}h(x_2,\ldots,x_{2'})\big) = (q^l[m_1']_q + q^{-m_1'}[l]_q)x_{1'}^{m_1'}h(x_2,\ldots,x_{2'})$$
$$= [m_1'+l]_q x_{1'}^{m_1'}h(x_2,\ldots,x_{2'}).$$

Therefore, returning to equation (34) we obtain

$$\partial_{1'}x_{1'}^{m_1'}h(x_2,\ldots,x_{2'}) = \big(q\hat{x}_{1'}\partial_{1'} + q^{m_1'-1+l} - (q-q^{-1})[m_1'-1+l]_q\big)x_{1'}^{m_1'-1}h(x_2,\ldots,x_{2'})$$
$$= (q\hat{x}_{1'}\partial_{1'} + q^{-m_1'-l+1})x_{1'}^{m_1'-1}h(x_2,\ldots,x_{2'}).$$

Applying this relation for $x_{1'}^{m_1'-1}h, x_{1'}^{m_1'-2}h,\ldots,x_{1'}h$ and lemma 1 we receive

$$\partial_{1'}x_{1'}^{m_1'}h(x_2,\ldots,x_{2'}) = (q^{1-m_1'-l} + q^{1-m_1'-l+2} + \cdots)x_{1'}^{m_1'-1}h(x_2,\ldots,x_{2'})$$
$$+ q^{m_1'}x_{1'}^{m_1'}\partial_{1'}h(x_2,\ldots,x_{2'}) = q^{-l}[m_1']_q x_{1'}^{m_1'-1}h(x_2,\ldots,x_{2'}).$$

Now using equation (33) we derive equation (32). The proposition is proven. □

Let $h_l \in \mathcal{H}_l^{(N-2)}$, where $\mathcal{H}_l^{(N-2)}$ is the space of $\Delta_q^{(N-2)}$-harmonic polynomials in $x_2, x_3, \ldots, x_{2'}$. Then $x_1^{m_1}x_{1'}^{m_1'}h_l \in \mathcal{A}_m$, $m = m_1 + m_1' + l$. Using formula (32), in the same way as in the case of the formula for $\Delta_q^k\big(x_1^{m_1}x_{1'}^{m_1'}\big)$ in section 6, we find that

$$\Delta_q^k\big(x_1^{m_1}x_{1'}^{m_1'}h_l\big) = (1+q^{N-2})^k q^{(m_1+m_1'-k)k}q^{-(N-2)k/2}\frac{[m_1]_q![m_1']_q!}{[m_1-k]_q![m_1'-k]_q!}x_1^{m_1-k}x_{1'}^{m_1'-k}h_l.$$

Now using formulae (21) and (23) we derive that $\mathsf{H}_m\big(x_1^{m_1}x_{1'}^{m_1'}h_l\big) = t_{m_1m_1'}^{N,m}h_l$, where

$$t_{m_1m_1'}^{N,m} = \sum_{k=0}^{\min(m_1,m_1')}\frac{(q^{-2m_1};q^2)_k(q^{-2m_1'};q^2)_k}{(q^2;q^2)_k(q^{-N-2m+4};q^2)_k}\frac{q^{(-N-l+3)k/2}}{(1+q^{N-2})^k}Q^k x_1^{m_1-k}x_{1'}^{m_1'-k}. \quad (35)$$

Using formula (15) the polynomials $t_{m_1m_1'}^{N,m}$ can be represented in a form similar to equations (26) and (27).

Proposition 6. *The space $\mathcal{H}_m$ can be represented as the orthogonal sum*

$$\mathcal{H}_m = \bigoplus\nolimits_{m_1,m_1'} t_{m_1m_1'}^{N,m}\mathcal{H}_{m-m_1-m_1'}^{(N-2)} \quad (36)$$

where $\mathcal{H}_{m-m_1-m_1'}^{(N-2)}$ is the space of $\Delta_q^{(N-2)}$-harmonic polynomials in $x_2, x_3, \ldots, x_{2'}$ and the summation is over all non-negative values of m_1 and m_1' such that $m - m_1 - m_1' \geqslant 0$.

Proof. The subspaces $t_{m_1m_1'}^{N,m}\mathcal{H}_{m-m_1-m_1'}^{(N-2)}$ from equation (36) do not pairwise intersect and elements from different subspaces are linearly independent. Therefore, on the right-hand side of equation (36) we have a direct sum. Besides, we have $\mathcal{H}_m \supseteq \bigoplus_{m_1,m_1'} t^{N,m}_{m_1 m_1'} \mathcal{H}^{(N-2)}_{m-m_1-m_1'}$. By direct computation (using corollary 2) we show that dimensions of the spaces on both sides of equation (36) are equal to each other. Now, in order to prove our proposition, we have to show that the sum on the right-hand side is orthogonal.

It is easy to prove that the subspaces on the right-hand side of equation (36) are eigenspaces of the operators $\hat{K}_i, i \leqslant n$, from formula (5) belonging to different eigenvalues. As in the proof of theorem 1, it follows from this fact that the sum (36) is orthogonal. The proposition is proven. □

Now we apply the decomposition (36) to the subspaces $\mathcal{H}^{(N-2)}_{m-m_1-m_1'}$ and obtain

$$\mathcal{H}_m = \bigoplus_{m_1,m_1'} \bigoplus_{m_2,m_2'} t^{N,m}_{m_1 m_1'} t^{N-2,l}_{m_2 m_2'} \mathcal{H}^{(N-4)}_{l-m_2-m_2'} \qquad l = m - m_1 - m_1' \tag{37}$$

where $\mathcal{H}^{(N-4)}_{l-m_2-m_2'}$ are the subspaces of homogeneous $\Delta_q^{(N-4)}$-harmonic polynomials in $x_3, x_4, \ldots, x_{3'}$. Continuing such decompositions we obtain the decomposition

$$\mathcal{H}_m = \bigoplus\nolimits_{\mathbf{m},\mathbf{m}',k} \mathbb{C}\,\Xi_{\mathbf{m},\mathbf{m}',k}(x_1, \ldots, x_{1'})$$

if $N = 2n$ and the decomposition

$$\mathcal{H}_m = \bigoplus\nolimits_{\mathbf{m},\mathbf{m}',\sigma} \mathbb{C}\,\Xi_{\mathbf{m},\mathbf{m}',\sigma}(x_1, \ldots, x_{1'})$$

if $N = 2n + 1$, where $\mathbf{m} = (m_1, m_2, \ldots, m_{n-1})$, $\mathbf{m}' = (m_1', m_2', \ldots, m_{n-1}')$ in the first case, $\mathbf{m} = (m_1, m_2, \ldots, m_n)$, $\mathbf{m}' = (m_1', m_2', \ldots, m_n')$ in the second case and m_j are non-negative integers, k take integral values, and $\sigma = 0$ or 1. The basis q-harmonic polynomials $\Xi_{\mathbf{m},\mathbf{m}',k}(x_1, \ldots, x_{1'})$ of $\mathcal{H}_m$ are given by the formula

$$\Xi_{\mathbf{m},\mathbf{m}',k}(x_1, \ldots, x_{1'}) = t^{N,m}_{m_1 m_1'} t^{N-2,m-m_1-m_1'}_{m_2 m_2'} \ldots t^{4,m-\sum_{i=1}^{n-2} m_i - \sum_{i=1}^{n-2} m_i'}_{m_{n-1} m_{n-1}'} t^{2,k} \tag{38}$$

if $N = 2n$ and by the formula

$$\Xi_{\mathbf{m},\mathbf{m}',\sigma}(x_1, \ldots, x_{1'}) = t^{N,m}_{m_1 m_1'} t^{N-2,m-m_1-m_1'}_{m_2 m_2'} \ldots t^{3,m-\sum_{i=1}^{n-1} m_i - \sum_{i=1}^{n-1} m_i'}_{m_n m_n'} x^{\sigma}_{n+1} \tag{39}$$

if $N = 2n + 1$. In equation (38), $t^{2,k} = x_n^k$ if $k > 0$, $t^{2,k} = 1$ if $k = 0$, and $t^{2,k} = x_{n'}^{-k}$ if $k < 0$. Note that the integers $k, \sigma, \mathbf{m} = (m_1, m_2, \ldots)$ and $\mathbf{m}' = (m_1', m_2', \ldots)$ take such values that

$$m_1 + m_1' + m_2 + m_2' + \cdots + m_{n-1} + m_{n-1}' + k = m \tag{40}$$

for $N = 2n$ and

$$m_1 + m_1' + m_2 + m_2' + \cdots + m_n + m_n' + \sigma = m \tag{41}$$

for $N = 2n + 1$. Besides, conditions such as the condition $m - m_1 - m_1' \geqslant 0$ of proposition 6 must be fulfilled at each step.

Theorem 3. *If $N = 2n$ then the polynomials (38), for which the equality (40) is satisfied, constitute an orthogonal basis of the space $\mathcal{H}_m$. If $N = 2n + 1$ then the polynomials (39), for which the equality (41) is satisfied, constitute an orthogonal basis of the space $\mathcal{H}_m$.*

Proof. The fact that the polynomials (38) for $N = 2n$ and the polynomials (39) for $N = 2n+1$ constitute a basis of $\mathcal{H}_m$ has been proved above. Orthogonality of basis elements is proved in the same method as in theorem 1. The theorem is proven. □

The polynomials (38) and (39) represent solutions of the equation $\Delta_q p = 0$ in separated coordinates. In the classical case, these polynomials are products of Jacobi polynomials; see chapter 10 of [1].

Note that if we were to consider zonal and associated spherical polynomials on the more complicated quantum spaces (for example, on the quantum Grassmannians) on which the quantum group $SO_q(N)$ acts, then, as we believe, they would be expressed in terms of orthogonal polynomials related to root systems (such as were considered in [10]).

It is interesting to have explicit formulae showing how the generators K_i, E_i, F_i of $U_q(\mathrm{so}_N)$ act on the basis elements of theorem 3. However, the derivation of these formulae is very awkward and the formulae are not simple. We shall consider them in a separate paper.

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